PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

LECTURE 3 WARM-UP: *TWO* TALL BUT NARROW WALLS

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Once again, we find ourselves as pedestrians on the Euclidean plane. This time, however, we are faced by an even more intimidating obstacle than before: *two* tall but narrow walls, placed directly in front and behind us.

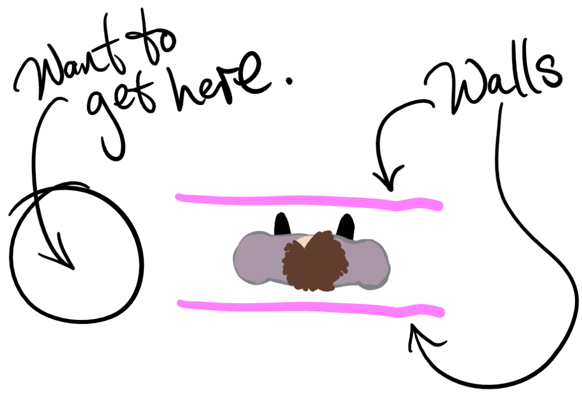

FIGURE 1. Pictorial depiction of the second wall puzzle

Our movement is quite restricted between the two walls: there is very little turning room, nor is there much room for moving forward or backward. Also, for completely justifiable reasons,[1] these are the only ways we can move; in order to get out from between the two walls, we must perform a sequence of motions whose velocities are scalar multiples of either $\omega_{\mathrm{I}(2)}^{-1}(e_1)$ or $\omega_{\mathrm{I}(2)}^{-1}(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right])$. The problem is similar, but not quite the same, as what one encounters when parallel parking a car.

There is basically only one thing we *can* do in this situation: we *wiggle* to the left. More explicitly, we start by turning a small amount counterclockwise, with velocity $\omega_{\mathrm{I}(2)}^{-1}(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right])$ for some small time ε, then

[1]You've temporarily forgotten how to move left, or something. The walls are magical? I don't know. Coming up with a good, intuitive reason for why we can't literally sidestep this problem was hard. Just go with it.

we move a little bit forward, with velocity $\omega_{\mathrm{I}(2)}^{-1}(e_1)$ for some small time δ, then we turn back clockwise, with velocity $-\omega_{\mathrm{I}(2)}^{-1}(\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right])$ for time ε, then finally moving backward, with velocity $-\omega_{\mathrm{I}(2)}^{-1}(e_1)$ for time δ. Repeating this sequence of motions enough times, we should move far enough left to escape the two walls.

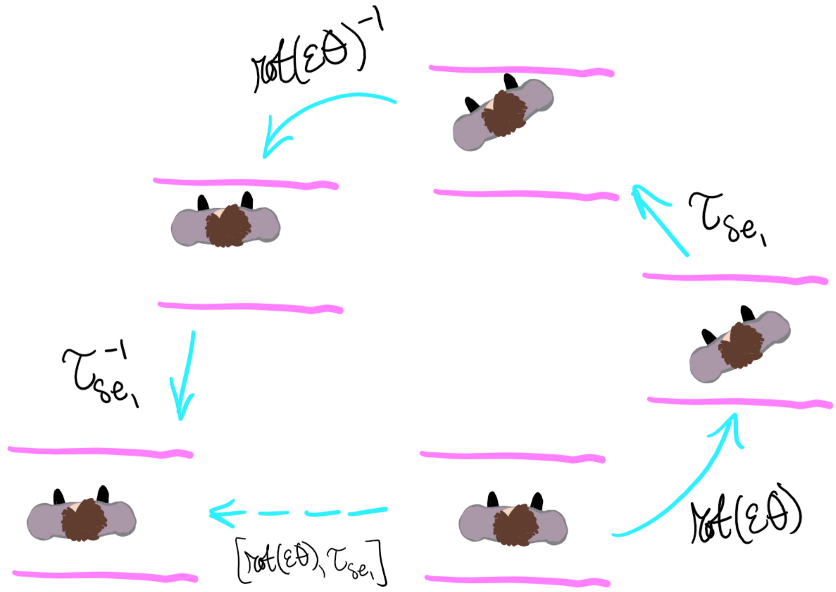

FIGURE 2. A depiction of one "cycle" of wiggling to the left

Note that, if we start at $g \in \mathrm{I}(2)$ and do one "cycle" of wiggling, then we will end up at $g(\mathrm{rot}(\varepsilon)\tau_{\delta e_1}\mathrm{rot}(\varepsilon)^{-1}\tau_{\delta e_1}^{-1})$. This element by which we end up right-translating has a specific name in group theory: it is the *commutator* $[\mathrm{rot}(\varepsilon), \tau_{\delta e_1}]$, where $[\cdot,\cdot]$ given by $[a,b] := aba^{-1}b^{-1}$ on elements of the Lie group.

Let us also notice something about the bracket operation on the Lie algebra. The bracket on $\mathfrak{i}(2) \approx \mathbb{R}^2 \ni \mathfrak{o}(2)$ is given by

$$[(v,X),(w,Y)] = (Xw - Yv, [X,Y]) = (Xw - Yv, 0),$$

so in particular,

$$[(0, \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]), (e_1, 0)] = (\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]e_1, 0) = (e_2, 0).$$

In other words, if we take the bracket of "unit positive angular velocity" with "unit forward velocity", then we get "unit leftward velocity".

All this to say, the bracket on the Lie algebra sort of just looks like "infinitesimal wiggling", and we can get a fairly good idea of what the bracket looks like by using the commutator on the Lie group. This can be formalized via what is sometimes known (see, for example, Proposition 3.4.7 and Proposition 9.2.14 of [1]) as Trotter's formula:

$$\exp([X,Y]) = \lim_{k\to+\infty}\left[\exp\left(\frac{1}{k}X\right),\ \exp\left(\frac{1}{k}Y\right)\right]^{k^2}.$$

Alternatively, we can get basically the same intuition from defining the bracket of the Lie algebra $\mathfrak{g}$ of a Lie group G by

$$[X,Y] := \omega_G([\omega_G^{-1}(X), \omega_G^{-1}(Y)]),$$

where the bracket on the right-hand side is the Lie bracket of vector fields, and then using the usual intuition for the Lie bracket in terms of closing up parallelograms.

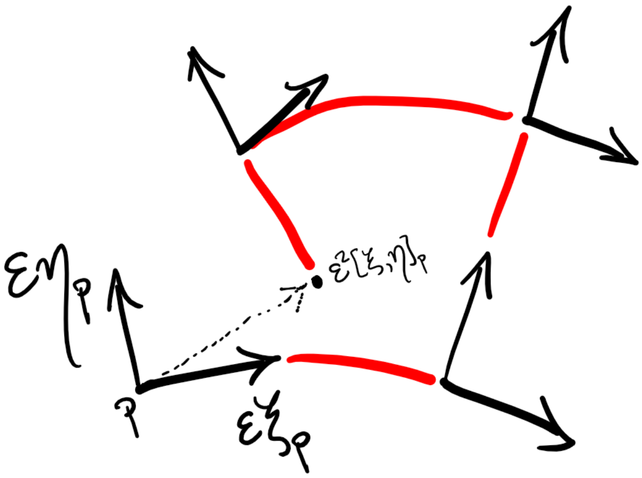

Figure 3. Take two vector fields ξ and η. When we flow for small time ε along ξ, then for time ε along η, then for time ε along $-\xi$, then finally for time ε along $-\eta$, we end up around where we'd get by flowing along the vector field $[\xi,\eta]$ for time ε^2

References

[1] Hilgert, J., Neeb, K.-H.: *Structure and Geometry of Lie Groups*. Springer Monographs in Mathematics, Springer-Verlag New York (2012)